Calculate the limit

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$$\lim_{n\to\infty}\frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\dots\binom{n}{n}}}{e^{n/2}n^{-1/2}}.$$

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Let $sf(n) := 1!2!\dots n!$ (superfactorial) and $H(n) := 1^1 2^2\dots n^n$ (hyperfactorial).

Since $sf(n) = 1^n 2^{n-1}\dots n^1 = \frac{(n!)^{n+1}}{H(n)}$ then $\prod_{k=1}^{n}\binom{n}{k} = \prod_{k=1}^{n}\frac{n!}{k!(n-k)!} =$

$\frac{(n!)^n}{sf(n)\cdot sf(n-1)} = \frac{(n!)^{n+1}}{sf^2(n)} = \frac{(n!)^{n+1}H^2(n)}{(n!)^{2(n+1)}} = \frac{H^2(n)}{(n!)^{n+1}}$ and, therefore,

$$\frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\dots\binom{n}{n}}}{e^{n/2}n^{-1/2}} = \frac{n^{1/2}}{n!\cdot e^{n/2}}\sqrt[n]{\frac{H^2(n)}{n!}}.$$

Using the following asymptotic equivalences

$\sqrt[n]{n!} \sim ne^{-1}$, $n! \sim n^n e^{-n}\sqrt{2\pi n}$ (Stirling approximation formula) and

$\sqrt[n]{H(n)} \sim e^{-\frac{n}{4}}\cdot n^{\frac{n+1}{2}}$ (see [1] or [2] or [3]).

we obtain $\frac{n^{1/2}}{n!\cdot e^{n/2}}\sqrt[n]{\frac{H^2(n)}{n!}} \sim \frac{n^{1/2}}{n^n e^{-n}\sqrt{2\pi n}\cdot e^{n/2}}\cdot\frac{e^{-\frac{n}{2}}\cdot n^{n+1}}{ne^{-1}} = \frac{e}{\sqrt{2\pi}}$.

Hence, $\lim_{n\to\infty}\frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\dots\binom{n}{n}}}{e^{n/2}n^{-1/2}} = \frac{e}{\sqrt{2\pi}}$

[1] G. Polya, G. Szego, Problems and Theorems in Analysis, Part 1, Problem 15, p. 50.

[2] Solutions in real analysis, Masayoshi Hata, p.2, Problem 1.3.

[3] Lemma in the solution to problem W8, J. Wildt IMC 2015 (Proposed by Arkady Alt) pp.431,432:

http://www.equationroom.com/Publications/Suggested%20problems%20(with%20solutions In%20OCTOGON%20(J.Wildt%20IMO)/W8%202015%20Solutions%20(Vol.23,%20n.2,% October%202015,%20p.429-437))%20%20.pdf